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13. ABSTRACT *(Maximum 200 words)*

The Workshop/School on Stochastic Partial Differential Equations: Theory and Applications was held January 3 through January 7, 1996 in Los Angeles. The event was organized and hosted by the University of Southern California. Funding was provided by ONR ($20,000), ARO ($5,000) and IMA ($5,000). There were 85 participants registered for the Workshop.

A series of mini-courses during the Workshop was given by world-renown experts in stochastic analysis and its applications including D. Dawson(Carleton University, Canada), G. Glimm (SUNY of Stony Brook), N. Krylov (University of Minnesota), and J. Lebowitz (Rutgers University). These courses served as a comprehensive review of the state of the art in SPDE's and their applications.

An agreement was reached between the organizers of the conference and the American Mathematical Society that the latter will publish a Proceedings of the Workshop in the form of a monograph. It is expected that the volume will be published in 1997.

The Workshop was a great success. It demonstrated clearly that SPDE's is one of the most dynamically developing areas of probability and statistics with a wide range of important applications. Many of them are of substantial importance for Navy applications (e.g. nonlinear filtering, prediction and smoothing, stochastic numerics, SPDE's of physical oceanography, etc.).

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Stochastic Partial Differential Equations: Six Perspectives A Proposal

R. Carmona B.L. Rozovsky
Editors

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Contents

Preface

The field of Stochastic Partial Differential Equations (SPDE's), one of the most dynamically developing areas of mathematics, lies at the cross section of probability, partial differential equations, and mathematical physics. It is especially attractive because of its interdisciplinary character and enormous richness of current and potential future applications.

Due to its spectacular success, the topic of SPDE's has gained much attention in the last two decades. However, as it often happens to research fields in their early stages of development, the paradigm of SPDE's is still fairly "soft": the same name SPDE covers different topics for different people.

In the world of probabilists (especially those working in the theory of stochastic processes), the field of SPDE's covers the analysis of partial differential equations driven by a random term which can be interpreted as a white noise in time, or a space-time white noise. The emphasis here is on stochastic analysis: existence and uniqueness problems, control of the fine structure of the solution when viewed as stochastic processes and/or random fields (regularity properties of the sample realizations, Markov properties, etc.).

In the world of biologists, physicists, fluid mechanicians, and more so in the world of most applied mathematicians, the field of SPDE's covers the analysis of partial differential equations in the case when some of the coefficients are random. The models are derived phenomenologically on physical grounds, and a statistical approach is taken to handle complex phenomena such as turbulence, chaos, microscopic behavior,.... Most of the efforts in this area are devoted to the development of renormalization theory in which the SPDE's take a simpler form, and can become linear and deterministic.

This, of course, does not exhaust the plurality of interests in the field. As of now, SPDE's appear to be an exciting mosaic of highly interconnected

topics that spins around stochastics and partial differential equations rather than a well-ordered, established field.

It is hard to believe that any single book can or should treat the enormously complex field of SPDE's from a unified point of view. Consequently, this book is a collection of six important topics in SPDE's presented from a different view by distinguished scientists actively working on SPDE's and related areas.

The book consists of three parts. The first part covers methods of solutions of SPDE's. The second part studies relations of SPDE's and interacting particle systems arising in physics and biological sciences. The third is dedicated to general problems of stochastic modeling, applications to continuum physics, fluid dynamics, and physical oceanography with an emphasis on the computational aspects of SPDE's. Each part consists of two chapters written by different authors. Every chapter is a comparatively self-contained review of new and existing results of a particular subfield of SPDE's or a related area.

Of course, this book could not possibly cover all or even the most important developments and problems of SPDE's. However, we believe that it will provide the interested reader with an informative snapshot of this rapidly developing area.

Part I

General Theory of SPDE's

Chapter 1

Analytical approach to stochastic partial differential equations

N.V. Krylov

In this chapter we present a systematic exposition of the analytical approach to SPDEs of the form

$$u(t) = u(0) + \int_0^t (\sum_{i,j=1}^{d} a^{ij} u_{x^i x^j} + \sum_{i=1}^{d} b^i u_{x^i} + cu + f)\, ds$$

$$+ \sum_{k=1}^{\infty} \int_0^t (\sum_{i=1}^{d} \sigma^{ik} u_{x^i} + \nu^k u + g^k)\, dw^k_s, \tag{1.1}$$

where $a, b, c, f, \sigma, \nu, g$ are given predictable functions of $(\omega, t, x) \in (\Omega, R_+, R^d)$ and w^k are independent Wiener processes and the last integral is understood as Itô's stochastic integral.

One of the important impetuses for the theory of SPDEs is the problem of nonlinear filtering of diffusion processes. The filtering problem (estimation of a "signal" by observing its mixture with a "noise") is one of classical problems in the statistics of random processes. It also belongs to a rare type of purely engineering problems that have a precise mathematical formulation and allows for a mathematically rigorous solution.

The first remarkable results in connection with filtering of stationary processes were obtained by Kolmogorov and Wiener. After the celebrated paper by Kalman and Bucy was published in 1961, the 1960s and 1970s witnessed a rapid development of filtering theory for systems whose dynamics could be described by Itô's stochastic differential equations. First results for general diffusion processes were summed up in the books by Liptser and Shiryayev and by Kallianpur. A systematic investigation of the existence and properties of the filtering density for general diffusion processes started with works Krylov and Rozovskii (1977), (1978). An account of the results obtained before 1990 can be found in the book by Rozovskii (1990).

These kind of equations arises not only in the filtering problem but also in other applications of probability theory, e.g., to genetics (Fleming–Viot equations), quantum mechanics, magneto-dynamics and so on (see, for instance, the book by Rozovskii (1990)). There is an extensive literature devoted to investigations of the linear equation (1.1) as well as its different nonlinear modifications. The general theory of these equations started with works by Pardoux (1975) and Krylov and Rozovskii (1977). Various aspects of the theory not reflected in the book by Rozovskii are studied by Da Prato (1983), Walsh (1986), Gyongy (1989), Dowson, Iscoe and Perkins (1989), Haussman and Pardoux (1989), Buckdahn and Pardoux (1990), Brzezniak (1991), Krylov and Gyongy (1992), Flandolli (1992), Mueller (1991).

So far the Sobolev-Hilbert spaces W_2^n were the spaces of choice to characterize the smoothness of the a posteriori density. Unfortunately, these spaces are not very convenient from the point of view of practical approximations of the solutions. The reason for this is that $W_2^n(R^d) \subset C^{n-d/2}(R^d)$ only if $2n > d$, and one can prove that the solutions belong to $W_2^n(R^d)$ only if the coefficients are $n-2$ times continuously differentiable. Therefore, if we want to get the solutions m times continuously differentiable with respect to $x \in R^d$, we have to suppose that the coefficients of the equation are more than $m + d/2 - 2$ times continuously differentiable even if the free terms are of class $C_0^\infty(R^d)$. At the same time $W_p^n(R^d) \subset C^{n-d/p}(R^d)$ if $pn > d$, and by taking p sufficiently large one sees that the solutions have almost as many usual derivatives as generalized ones. Actually, exactly for this purpose the spaces $W_p^n(R^d)$ with $p \geq 2$ have already been used in the SPDE theory (see, for instance, Rozovskii (1990), Krylov and Rozovskii (1977)), but the corresponding results obtained by integration by parts were far from being sharp.

Another advantage of the W_p^n setting with $p \geq 2$ can be seen in the case of very popular equations with so-called cylindrical white noise (see, for instance, Mueller (1991), Nualart and Pardoux (1992), Walsh (1986) and references therein). These equations can be included in the general W_p^n-theory as particular examples for any $p \geq 2$. However, in the case $p = 2$ the general theory gives only integrability of all powers of the solution. On the other hand, for $p > 2$, the general results imply continuity of the solution. By the way, it is essential to work with the whole scale of values of n: $n \in (-\infty, \infty)$, and with the spaces of Bessel potentials $H_p^n(R^d)$. In the case of equations with the cylindrical white noise we need n slightly less than $(-3/2)$.

Here we concentrate on H_p^n–theory for all range of n, p and give a self contained exposition of the theory in *the whole space.* In particular, we obtain an existence theorem which extends corresponding known results (see Walsh (1986)) to equations with *random and variable* coefficients, which was just inconceivable under usual treatment of these equations. We also give a short proof of a *generalization* of an important result from Mueller (1991) concerning the nonexplosion of solutions of a nonlinear SPDE.

In addition, we also present some elements of the theory for smooth domains. There are some unusual difficulties which *require* introducing *weighted Sobolev spaces.*

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Chapter 2

Martingale Problems for Parabolic SPDE's and Absolute Continuity of their Solutions

R. Mikulevicius and B.L. Rozovskii

2.1 Introduction

Loosely speaking, all exisiting methods of solving parabolic SPDE's can be split into two categories. Below we will refer to them as pathwise and statistical, respectively.

The distinction between these two approaches stems from a fundamental difference in the interpretation of the notion of a solution. The pathwise approach interprets a solution as a random function of time (path), taking values in some functional space Ξ, which satisfies the equation in question for any (or almost any) given realization of the random factors involved in this equation.

On the contrary, the statistical solution is a probability law on the space of paths with value in Ξ, so that all the paths from the support of this measure solve our equation. (In which sense a path satisfies the equation might vary in both approaches). Of course the probability law of a pathwise solution

is a statistical solution for the same equation. So the statistical solution is an extension of the notion of a pathwise solution. That is why the former is sometimes referred to as weak and the latter as strong.

The roots of most pathwise methods can be traced to deterministic theory of differential equations. In particular, this applies to theories of linear SPDE's and SPDE's of monotone type (see e.g. DaPrato, Zabchick [17], Krylov [this volume], Krylov, Rozovskii [36], Pardoux [54], Rozovskii [57]). On the contrary the statistical methods essentially have no peers in the deterministic theory of PDE's. This approach originated in the theory of ordinary SDE and has proven to be a most effective tool in studies of nonlinear equations. Statistical approach to stochastic PDE was championed by Skorohod [58] (weak solutions) and Strook and Varadhan [60] (martingale problems). Viot [62] pioneered applications of statistical solutions to SPDE's, specifically he studied martingale problems for stochastic Navier-Stokes equation and Fleming-Viot super process. Later this approach to SPDE's was developed by Grigelionis, Mikulevicius [26], Kozlov [33], Metivier [49], Mikulevicius, Rozovskii [50] and many others. In particular, this approach has proven to be very helpful in studying super-Brownian motion and related branching processes (see Dawson, Perkins [this volume] and references therein).

A review of the pathwise methods of solving linear and quasi-linear parabolic SPDE's is given in Chapter 2. The present Chapter deals exclusively with statistical solutions of nonlinear parabolic SPDE's. More specifically we restrict ourselves to martingale problems for parabolic SPDE's. This setting introduced by Stroock and Varadhan [60] is probably the most flexible and general form of the statistical approach.

2.2 Stochastic Analysis in Topological Vector Spaces

We begin with some elements of infinite-dimensional stochastic analysis to set up a general framework for the future.

Stochastic integration in infinite dimensional spaces is a mature area. Several important classes of stochastic integrals were introduced and studied in depth by Kunita [38], Metivier and Pistone [47], Meyer [46], Metivier and Pellaumail [48], Gyöngi and Krylov [28], Grigelionis and Mikulevicius

[26], Walsh [63], Korezlioglu [32], Kunita [37], etc. Not surprisingly, the approaches to infinite dimensional stochastic integration proposed in these works have some similarities but also some distinct features. The latter are mainly related to the specifics of the spaces and processes involved. For example, the integral with respect to a stochastic flow (see Kunita [37], and also Gihman, Skorohod [25]) and the integrals with respect to orthogonal martingale measures (see Gyöngy, Krylov [28], Walsh [63]) seem to have very little in common. In fact, the relation between these two integrals as well as others mentioned above is stronger then it might appear. More specifically, it will be shown in this section that all these integrals and some other are particular cases of one stochastic integral with respect to a local square integrable cylindrical martingale in a topological vector space.

2.3 Infinite Dimensional Martingale Problem. Existence of Solutions

The powerful idea to prove existence of a weak solution to an ordinary Itô equation (or the corresponding martingale problem) using weak compactness of measures generated by a sequence of approximate solutions was championed by Skorohod[58] and Stroock and Varadhan[60]. It turned out that this technique can be extended to infinite-dimensional stochastic equations including important classes of non-linear SPDE's (see e.g. Viot[62], Grigelionis and Mikulevicius[26] etc.).

In this section we study the martingale problem for an abstract nonlinear stochastic PDE in the Gelfand triple $V \subset H \subset V'$ where H is a separable Hilbert space and V and its dual V' are reflexive Banach spaces. It is assumed that the SPDE is coercive (in the sense of Krylov, Rozovskii (1981) and its operator coefficients are weakly continuous. We will prove the existence theorem in three steps. Firstly, by using coercivity assumption we will establish a priori estimates. Secondly, these estimates will be used to establish tightness of distributions generated by approximate solutions of our equation. Finally we will use Prokhorov theorem to establish weak convergence of the approximate weak solutions and then will pass to the limit.

It should be noted that the study of weak solutions of SPDE's requires consideration of spaces and trajectories endowed with topologies which are

not complete separable metrizable topologies. The topology of interest in each particular case is commanded by the functionals of the paths which are to be studied. The topology has to be fine enough to make given functionals continuous, but not too fine if one wants a given sequence of measures to be weakly compact.

In the end of this chapter we will consider two SPDE's of particular interests: the equation for super Brownian motion on R^d and the equation of stochastic quantization in $P(\varphi)_2$ theory (see e.g. Jona-Lasinio, Mitter[30] (1985)).

2.4 Absolute Continuity and Uniqueness of Weak Solutions

The methodology developed to study uniqueness of weak solutions of finite-dimensional stochastic ODE's appears to be difficult to apply to SPDE's. It rests on analytical results for the Kolmogorov equtaion (or its elliptic counterpart) related to the martingale problem in question. To apply the same idea to SPDE's, one would need similar results for (variational) parabolic or elliptical equations on infinite dimensional domains. Unfortunately, such results in many interesting cases are not known at the moment and seem to be hard to obtain.

To overcome this difficulty we take a different approach. It applies to cases when a weak solution is absolutely continuous with respect to a measure which itself is a weak solution to a simpler reference equation. Roughly speaking, we will prove that if a weak solution μ to some SPDE is locally unique, then all other measures absolutely continuous with respect to μ are also unique weak solutions to SPDE's which differ from the original one by additive perturbations of its "drift operator."

By no means this approach is new; it has been routinely used in finite-dimensional stochastic ODE's. To apply this method to SPDE's, we will first study the problem of absolute continuity of solutions of infinite-dimensional martingale problem. Specifically, we will prove a necessary and sufficient criterion for the above property of martingale problems.

This result generalizes the well known criterion for absolute continuity of measures generated by finite-dimensional diffusion processes (see Liptser-

Shiryaev[43]) and its infinite-dimensional version (Kozlov[33]).

In the second part of this section we will apply the aforementioned results to various SPDE's.

Specifically, we will prove the uniqueness of the equation of stochastic quantization

$$\begin{aligned} dx_t &= \left[\tfrac{1}{2}\Delta X_t - : x_t^3 :\right] dt + dW_t \\ x_0 &\sim c_1 \exp\{-c : x^3 :\} \mu(dx) , \end{aligned}$$

where W_t is a space time White noise on R^2 and $\mu(dx)$ is a Gaussian measure on $L_2(R^2)$ with zero mean and covariance Δ^{-1}. We will also prove that the probability distribution generated by this equation is singular to the one generated by the Ornstein-Uhlenbeck equation

$$\begin{aligned} dX_t &= \tfrac{1}{2}\Delta X_t dt + dW_t \\ X_0 &\sim c_1 \exp\{-c : x^3 :\} \mu(dX) . \end{aligned}$$

In the conclusion of this section we will give an explicit characterization of all measures absolutely continuous with respect to the probability distribution of the Super-Brownian motion.

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Part II

SPDE's and Interacting Particle Systems

Chapter 3

Limits of Renormalized Branching Particle Systems

D.A. Dawson and E.A. Perkins

3.1 Measure-valued processes

This chapter provides a brief self-contained introduction to branching particle systems and super-Brownian motion. We begin with a description of a general family of particle systems in the setting of measure-valued martingale problems. Then the special case of branching Brownian motions is given and then super-Brownian motion is obtained as a renormalized limit of such processes. In $\mathbb{R}^1$ super-Brownian motion has a density process which satisfies a stochastic partial differential equation but in higher dimensions the processes have singular measure states and for this reason the spde interpretation must be replaced by the martingale problem. Several basic tools will be introduced including weak convergence, measure-valued martingale problems and the Poisson cluster representation. Finally a number of extensions, examples and applications of measure-valued branching systems are presented. A tentative list of section headings is:

- A General Class of Particle Systems
- Super-Brownian motion as the limit of branching Brownian Motion

- The Poisson Cluster Representation
- Relations with SPDE
- Super-random walk on Z
- The spde in $\mathbb{R}$
- Weak convergence approach in $\mathbb{R}$
- Extension to non-square integrable branching and infinite measures
- Branching at a single point catalyst
- Lattice trees and integrated super-excursion
- The Fleming-Viot process as conditioned SBM

3.2 History, Genealogy

Branching particles systems have a rich family structure which is not explicit in the characterization of super-Brownian motion given in Chapter 1 but which is crucial for a deeper study of this process. In this chapter we introduce the genealogy first at the particle level and then derive historical Brownian motion which is an enriched version of super-Brownian motion. In this context one can obtain an analogue of Lévy's classical modulus of continuity of Brownian motion which serves as an important tool in studying the fine structure of super-Brownian motion. The sections will include:

- Historical Brownian motion
- Historical Modulus of continuity

3.3 Small Scale Behavior

In general the subject of measure-valued processes is still in an early stage of development. However the study of the local structure of super-Brownian motion is now quite highly developed. In this section we give a survey of results on the small scale properties of super-Brownian motion in $\mathbb{R}^d$ and include some detailed proofs to illustrate the methodologies which have been developed in this area. This includes results on the closed support of SBM, occupation measures and local times, polar sets, collisions of closed supports of SBM and closed supports with Brownian particles. A tentative list of section headings include:

- Structure of the closed support of SBM
- Occupation measures and local times
- Range and multiple points of SBM
- Hitting Probabilities
- R and G polar sets
- Collisions of closed supports

3.4 Some General Formulations of Measure-valued Branching Processes

In this chapter we briefly review (without proofs) Dynkin's program to characterize the general class of measure-valued branching processes in terms of natural characteristics. We also consider the characterization of these processes in terms of a family of martingale problems.

- Dynkin's formulation of a general class of MVB processes
- The martingale problem formulation

3.5 Large Scale Behavior

In this section we give an introduction to the large scale properties of branching Brownian motions and super-Brownain motion in $\mathbb{R}^d$. This involves a description of the competition between the "clumping" produced by the branching mechanism and the spatial homogenization produced by the migration mechanism. Two basic tools are used in order to gain an understanding of these problems - the first is the historical process which allow the description of the large space-time scale behavior of surviving families (clans) which in turn lead to a detailed description of the equilibrium states in high dimensions and clumping structure in critical and subcritical dimensions. We also briefly describe the dyanamics of the infinite clans which arise in high dimensions. Finally we briefly describe some recent first steps in a program to establish that the large space-time scale behaviors exhibited by critical branching systems is shared by a larger "universality class" of spatially homogeneous measure-valued systems.

- The persistence-extinction dichotomy
- Clumping in low dimensions
- Ergodic behavior
- The equilibrium clan decomposition
- Clan dynamics
- Universality of the extinction-persistence dichotomy
- Thermodynamic limit approach to super-random walk - the hierarchical mean field limit (if space permits)

3.6 Interactive Branching Systems

This chapter begins with an informal discussion of some potential applications of measure-valued processes and the interaction mechanisms which arise naturally in these applications. The rigorous study of such systems is a subject of a lot of current research but many problems remain open. We describe

the formal structure of the martingale problems which arise and survey some of the classes of interactive measure-valued processes for which results have been obtained. One of these is the continuous analogue of a branching particle system in which the motions of the individual particles are influenced by the other particles which are present. Another is the class of processes which arises in population genetics as an idealization of branching systems in an environment with locally finite carrying capacities - these are the stepping stone systems of Fleming-Viot processes.

- Applications of measure-valued processes
  - reaction diffusion systems, diffusion of innovations, population genetics, evolutionary theory, ecology, distributed and hierarchically structured information systems
  - Describe the models, interactions involved and problems posed.
- An overview of interactions for particle systems and measure-valued processes - structure of formal generators.
- Examples of branching with interaction:
  - State dependent mean offspring size
  - Zero range interactive killing
  - Particle medium and particle-medium-particle interactive motion
  - Interactive branching in $\mathbb{R}$
  - Catalytic branching
  - Mutually catalytic branching
  - Multilevel branching
- Fleming-Viot processes and stepping stone models
  - the infinitely many types stepping stone model
  - selection and mutation
  - interactive sampling

Chapter 4

From Microscopic Dynamics To PDE's: Hydrodynamics And Fluctuations.

G. Giacomin and J.L. Lebowitz

4.1 Motivations And Overview

The hierarchical structure of nature often allows us to study the different levels of its hierarchy independently. A remarkable example of this fact is the success of hydrodynamic theories, which were and are often introduced with little or even no knowledge of the underlying molecular structure. At a more fundamental level, if we are focusing on atoms or molecules, considering the nuclei simply as point charges will often do little or no harm. Obviously the different levels are not completely independent and no sharp demarkation line can be drawn. In particular the behavior of a *rougher* or *larger scale* level should, at least in principle, be entirely derivable from a *more detailed* or *smaller scale* one. The apparent independence between different levels is a very deep fact: the large scale behavior often obeys macroscopic laws which have a very mild dependence on the details of the underlying finer level. These laws have generally the form of PDE's

$$\partial_t \mathbf{M}(\mathbf{r}, t) = \mathbf{F}(\mathbf{M})(\mathbf{r}, t) \tag{4.1}$$

in which $\mathbf{M}$ is a set of macroscopic variables (density, magnetization,...) depending on space and time. $\mathbf{F}$ is a functional which may depend on the value of $\mathbf{M}$ and its derivatives at $(\mathbf{r}, t)$ (local dependence), as well as on their values at different points in space and time (nonlocal dependence). The heat equation and the Euler and Navier–Stokes equations are examples of (1).

Several important questions connected with the interplay of different levels do arise. This is particularly relevant in situations in which a purely hydrodynamic description fails (e.g. appeareance of singularities in the macroscopic equations, nonuniqueness,...) or in understanding the small deviations from the hydrodynamic behavior, which, for example, may become crucial in presence of instabilities or in cases in which the system is observed over a very long period of time. A *two-level* version of (1) is

$$\partial_t \mathbf{M}^\epsilon(\mathbf{r}, t) = \mathbf{F}^\epsilon(\mathbf{M}^\epsilon)(\mathbf{r}, t) \tag{4.2}$$

in which $\mathbf{F}^\epsilon$ is dependent on a small parameter ϵ, representing the small contribution of the finer scale, and in the limit as ϵ vanishes $\mathbf{F}^\epsilon$ approaches $\mathbf{F}$. An often encountered choice of $\mathbf{F}^\epsilon$ is $\mathbf{F}^\epsilon = \mathbf{F} + \epsilon W(\mathbf{r}, t)$, in which $W(\mathbf{r}, t)$ is a stochastic term. A common problem is then the fact that $W(\mathbf{r}, t)$, being a contribution from a finer scale, may vary on spatial and temporal scales which are much shorter than the macroscopic scale. In other words $W(\mathbf{r}, t)$ may have very little regularity properties which can make (2) ill–posed.

We will focus on some mathematical models in which the transition from microscopic to macroscopic can be investigated in great detail. In particular, macroscopic laws of the type (1) can be rigorously established in this framework. The correction to the approximate limit behavior (1) can be often pinned down and one can give a precise meaning to equation (2). We will mainly confine ourselves to *phase segregation* models, which are particularly good examples, since various length scales naturally come up, as it will be explained below.

4.2 Macroscopic Phenomena and their Mathematical Description.

We will concentrate on phenomena which can be described in terms of density fields $\rho = (\rho_\alpha(\mathbf{r}, t))_{\alpha=1,\dots,n}$ (n is the number of components of the system) in which $\mathbf{r} \in \mathbf{R}^d$ (or a subset of it) and $t \in \mathbf{R}^+$. The dynamics may

drive the system to a homogeneous equilibrium (diffusive behavior) or it may lead to aggregation of the different components (phase segregation and pattern formation). There are essentially two mechanisms: one is the diffusion (which drives toward a homogeneous state) and the other one is the reaction (which introduces an interaction between the different components). We will introduce and briefly discuss nondegenerate diffusion equations, Reaction–Diffusion systems and more particular phase segregation models such as the Cahn–Allen (C–A) and Cahn–Hilliard (C–H) models. In particular the stress will be on two facts:

1. Phase separation models like Cahn-Allen and Cahn-Hilliard are already large scale dynamics, but it is on an even larger scale that the phase separation phenomena arise sharply and can be properly described;

2. Some very important phase separation phenomena are intrinsically non deterministic and it becomes vital to consider more than one level at the same time. Very common examples of these are the *quenching* phenomena, that is the onset of spatial patterns from a homogeneous state due to the sudden change of the status of a system from diffusive (high temperature) to phase segregating (low temperature).

4.3 Microscopic Structure and Its Mathematical Description.

The importance of simple models is well established for equilibrium behavior, that is for cases in which the system has reached its stationary state (this state is described in terms of a measure, the Gibbs measure, on the configuration space). The *Ising model*, the most simple of the non trivial Gibbs systems, has been successfully employed to model diverse phenomena and played a fundamental role in understanding the nature of *phase transition* phenomena, both from the applied and the mathematical viewpoint.

Up to now no model has acquired the status of prototype model for nonequilibrium phenomena. This is due in part to the much greater complexity of nonequilibrium: even the solutions of the macroscopic equations are far from being understood in the interesting cases. Nevertheless some computer simulations have shown that very simple models capture the essence of large

scale phenomena. Conforted by this empirical observation, we will mainly direct our attention to lattice systems with stochastic dynamics, the so called Interacting Particle Systems (IPS) [42, 59]. They are dynamical versions of the Gibbsian equilibrium models, in the sense that their invariant measure (in finite volume) is a Gibbs measure. The dynamics is Markovian and the configuration is updated at random (Poissonian) times according some local rules.

The IPS class is divided into two main classes: the Glauber (or *spin–flip*) dynamics, which does not conserve the sum of the occupation variables, and the Kawasaki (or *exchange*) dynamics, which has a conservation law. A third case is given by a superposition of Glauber and Kawasaki and it is particularly suitable to model Reaction–Diffusion systems.

4.4 From Microscopic to Macroscopic.

We will review some of the recent results on the derivation of hydrodynamic equations and we will focus mainly on two aspects:

1. the difference between high temperature (diffusive) behavior and low temperature (phase segregation) behavior, enphasizing the role of phase transitions;

2. the corrections to these hydrodynamic limits, which may be stochastic or deterministic, according to the cases. The corrections can be either small fluctuations around the limit or deviation from the limit behavior on long times.

Standard fluctuations results are (infinite dimensional) Central Limit Theorems: the limit is a Gaussian process, that is the solution of a linear SPDE. Nonlinear fluctuations can be viewed as a breakdown of the Central Limit Theorem and they appear only in particular (*critical*) instances: these are, of course, much harder to investigate.

4.5 Long Range Potentials: hydrodynamics, fluctuations and applications.

The program outlined in Section 3 (above) has been successfully carried out mostly for high temperature situations. At low temperatures the phenomena become much more complex and to gain a better understanding we will consider *local mean field models*, i.e. models in which the range of interaction is sent to infinity in a suitable way. More precisely, the 2–body interaction is given by $\gamma^d J(\gamma(x-y))$, in which x and y are two lattice sites, J is a smooth compactly supported function and γ is a positive real number. The parameter γ is sent to 0 as we consider the system in larger and larger boxes (Thermodynamical limit). The parameter γ introduces an extra level in the hierarchy of the system and allows us to analyze several phenomena in a rigorous fashion.

We will review some equilibrium results, starting with the original works [31, 39], including some results about metastability [55], and more recent ones.

Much work has been already done in the Glauber case (see [21] and references therein), where hydrodynamics, fluctuations, pattern formation and evolution have been investigated in detail. In this case even a nonlinear fluctuation regime has been identified.

We will consider in detail the dynamics in the Kawasaki case. Topics will include:

1. The hydrodynamic limit, given by a nonlocal evolution law (this reflects the nonlocality of the underlying model);

2. The connection of this macroscopic limit with the standard macroscopic model for the conserved case (C-H equation);

3. A Central Limit Theorem (fluctuations) for this model (original work);

4. Discussion of the relevance of the fluctuations for the problem of pattern formation;

5. Dynamical metastability;

6. Is there a nonlinear fluctuation regime? We will present an heuristic argument aimed at showing that a Stochastic Cahn–Hilliard equation is the scaling limit of a suitable fluctuation field at the critical point for the particle model under consideration.

Part III

Stochastic Modeling and Numerical Methods

Chapter 5

Stochastic Partial Differential Equations: Selected Applications in Continuum Physics

James Glimm and David Sharp

The central content and purpose of this chapter is to explain the scientific necessity for stochastic models, and certain key mathematical and theoretical issues which need to be addressed in this area in order to achieve the scientific goals proposed here. These two ideas occupy the first two sections of this paper. In the next two sections of this chapter, we illustrate these ideas by examining them from the point of view of fluid mixing. Here the discussion becomes more specific, and is based on properties of solutions of the Euler and Navier-Stokes equations, Darcy's law, the Buckley-Leverett equations and related equations. Because the scientific issues which we examine concern deep interactions between the nonlinear and stochastic aspects of the behavior of solutions of partial differential equations, neither of which is presently in a definitive or final stage, we appeal to a variety of methods for scientific understanding: theoretical analysis, numerical simulations, and analysis of experimental and field data.

Stochastic modeling occupies a central place in the derivation of the equations of continuum physics. This fact is obscured by another fact: the desired outcome of this modeling is often a deterministic partial differential equation. Thus the stochastic model may be important either as an intermediate step towards a deterministic theory, or as a final theory in its own right. As we move toward more realistic acceptance of probability as a final outcome for technology, such as in probabilistic risk assessment, probabilistic weather forecasts, probabilistic simulation studies to support petroleum reservoir management decisions, and probabilistic financial decision making, the role of the stochastic model as a final outcome will increase. As we demand increased accuracy and reliability from our deterministic models, the need to examine their stochastic components and derivations will also increase. For both reasons, we see a need to emphasize the importance of stochastic models of physical systems. The scientific issues raised in this article provide a selection of the type of questions which, in the authors' judgement, deserve deeper examination.

We emphasize dynamical instability and sensitive dependence on initial conditions, in a microscopic dynamics which can be deterministic in principle, as the origin of stochasticity in models of physics. Related is the absence of sufficient deterministic data, which probability models supply statistically. But the deterministic laws of physics cannot be neglected, so that the model will be stochastic, and not purely statistical. Moreover, partial data cannot be ignored, so that the stochastic probability measures will be conditioned by partial knowledge.

In section two, we will present some of the methods of a mathematical or analytic nature which have been useful in the study of stochastic modeling in application to continuum physics. We do not claim that the mathematical basis for these methods is well developed. To the contrary, the methods presented here are, for the most part, far less well developed mathematically than such methods as Ito integrals, martingales, or Wiener measure. In general, the lower level of development of the mathematical foundations is due to the greater difficulty presented by random fields as opposed to random processes. From a formal point of view, these two are related as partial differential equations are related to ordinary differential equations; alternately, we can say that the stochastic aspect of the problem converts a (deterministic) partial differential equation into such an equation in an infinite dimensional space. Thus the passage from deterministic to stochastic equations is comparable in

difficulty to the passage from ordinary to partial differential equations. The mathematical tools we discuss include random fields and ensemble averages, moment expansions and effective equations, renormalization group methods, and direct numerical simulations.

Section three is a survey of some very striking new results in the theory of fluid mixing layers, for compressible fluid dynamics. A body of work is presented, including, but not limited to, that of the authors and co-workers. Some of the principle results include scaling laws to describe mixing behavior, the first correct numerical simulations (as evidenced by agreement with experiment) for some typical fluid mixing problems, novel theoretical understanding, improved asymptotic expansions, newly derived equations, and the explicit construction of a renormalization group fixed point to explain scaling behavior.

Consider two fluids separated by an interface which is unstable relative to the flow pattern. As time progresses, the interface will become highly convoluted, and portions of each fluid will become entrained in the other, giving rise to a two-fluid mixing layer. The flow in the mixing layer has sensitive dependence on data, and thus requires stochastic modeling, even though the microscopic equations are deterministic. Shear layers (for which the tangential component of the velocity is discontinuous) are unstable according to the Kelvin-Helmholtz instability, leading to vortex formation and rollup of the interface. Jets are subject to breakup and atomization in a variety of regimes, governed by the jet velocity relative to the ambient medium, among other factors. Acceleration driven interfaces may also be unstable, if the fluids have different densities, so that the forces of acceleration on the two fluids are not equal. We consider here two cases of acceleration driven instabilities, the Rayleigh-Taylor (RT) instability associated with a steady acceleration, such as gravity, and the Richtmyer-Meshkov (RM) instability caused by impulsive acceleration, such as by a shock wave.

The quantities of interest are the size of the mixing region, as a function of time, and the statistical composition of the mixture, to give, for example the volume or mass average of each fluid, velocities, and mixing length scales. We use two theoretical methods to extract this information: a statistical analysis of elementary modes and closure of averaged equations with a statistical validation of the closure hypothesis. We also use direct numerical simulation to gather data to test these theories, and we analyze laboratory data for the same reason.

Section four has a similar goal, for the problem of mixing zones associated with flow in porous media. The results, while promising, are not as complete as those presented in section three. We consider both linear and nonlinear flow equations. Even the linear flow equations pose difficult problems, in that the random fields enter as equation coefficients. Thus the solution depends nonlinearly on the stochastic data. Standard Gaussian methods do not apply, and even if the data are Gaussian, the solution is non Gaussian.

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Chapter 6

Transport Simulations with 2-D Incompressible OU Velocity Fields

René A. Carmona

Our motivation comes from the analysis of the mathematical models of physical oceanography. If we consider for example the case of the temperature or the salinity of the ocean, these concentrations appear as solutions of linear parabolic equations driven by a the velocity field. We do not attempt to solve the nonlinear equation giving the velocity. Instead we use the known chaotic behavior of the solutions of the Navier-Stokes equation and real data from observations of drifters at the surface of the ocean to justify the choice of statistics for random models for the flows.

From the numerical point of view, the concentration SPDE's can be solved numerically by including a sample realization of the random coefficients (velocity and/or forcing term in the case of interest) and using standard deterministic methods such as finite elements, multigrid, $\cdots$ and the like. Such an approach is very involved and can be cumbersome. See [24] and [64] for recent attempts in the one-dimensional case with white noise models. Also, it lacks the insight provided by a fine analysis of the properties of the randomness in the system given by the Lagrangian analysis of the flow. This is the point of view taken in this paper. Since our goal is to simulate numerically

transport properties of random velocity fields, this Lagrangian approach is more natural than the Eulerian approach tied to the SPDE.

We first present real data from drifter motions at the surface of the ocean and we discuss briefly some of the statistical issues related to the spectral analysis of these data. Next, we present the theoretical framework of the theory of random velocity fields with Gaussian statistics, we formulate precisely the mathematical assumptions of the physical models most widely accepted (incompressibility, Kolmogorov spectrum, ...) and we address the simulations issues. Then, we consider the stationary models for which the random velocity field is independent of time. We review the recent results of Majda et al reported in [44], [45] and [23]. In the following, section we review the massively parallel simulations reported in [9], the conjectures formulated and illustrated in the paper and the theoretical results proved subsequently. In particular, we discuss the proof of the positivity of the upper Lyapunov exponent of the Jacobian flow given in [10] and the proof of the homogenization of the Lagrangian trajectories given in [12]. Finally, we discuss our work in progress [13] on the numerical estimates of the time evolution of the fractal dimension (Minkowski dimension to be specific) of the boundary of blobs of passive tracers transported by the flow and on the attempts to make the tracers feel the spectral singularity in the spectrum. The last section is devoted to the discussion of an alternative model of random velocity field (see [14] and [15]: The time stucture is still of the Ornstein-Uhlenbeck type while the spatial structure depends upon a random vorticity constructed from a spatial Poisson point process.

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